# DESIGN AND CONTROL

# OF

# A PERISHABLE INVENTORY MODEL

**Final Report of the Minor Research Project**

(MRP(S)-1408/11-12/KLM G002/UGC-SWRO) (XI plan) dated 10 July 2012 and 28 September 2012.)

**Submitted By**

**Dr. Varghese C. Joshua**

**Associate Professor,**

**Department of Mathematics,**

**CMS College, Kottayam, Kerala, 686001**

## Declaration

I, Dr. Varghese C. Joshua, associate Professor, Department of Mathematics, CMS College, Kottayam, do hereby declare that the project report entitled DESIGN AND CONTROL OF A PERISHABLE INVENTORY MODEL is the final report of the minor research project (No. MRP(S)-1408/11-12/KLM G002/UGC-SWRO) (XI plan) dated 10 July 2012 and 28 September 2012) carried out by me.

Dr. Varghese C. Joshua

Principal,

CMS College, Kottayam.

# Contents

CHAPTER I

# Introduction

*The most common misunderstanding about science is that scientists seek and find truth. They don't – they make and test models… Making sense of anything means making models that can predict outcomes and accommodate observations. Truth is Model.*

*Neil Gershenfeld, American Physicist, 2011*

One of the most powerful uses of mathematics is the mathematical modeling of real life situations. Mathematics can be used to adequately represent and in fact model the world, given that it displays a kind of exactness and necessity that appears to be in sharp contrast with the contingent character of reality. Wigner [30] famously claimed that the "miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.". Wigner emphasis the unexpected applicability of mathematics in natural sciences. He gives a large number of examples of effectiveness of mathematical models in natural sciences. He argues that the "miracle of the appropriateness of the language of mathematics for the formulation of the laws of nature is a wonderful gift which we neither understand nor deserve." It is a miracle that mathematical concepts introduced for aesthetic reasons turn out to unexpectedly apply.

Models describe our beliefs about how the world around us functions. In mathematical modelling, we try to translate those beliefs into the language of mathematics. Mathematics helps us to formulate ideas and identify underlying assumption as it is a precise language. Mathematics is also a concise language, with well-defined rules for manipulations. Moreover, all the results that mathematicians have proved over thousands of years are at our disposal. In the modern age, computers can be effectively used to perform numerical calculations. So in recent years, the use of mathematical models in research of science science have been given serious consideration by scientists.

**Objectives of the Project**

The purpose of the project is to study a perishable inventory model with Markovian Arrival Process input. We propose to use the performance measures thus obtained to control and design such models if closed form solutions are available. If the models is not analytically tractable we propose to develop an algorithmic solution using the set of tools in " matrix geometric method". FORTRAN code is proposed for the performance analysis.

We propose to study the following models:

First, we analyze an inventory model in which there are demands for "new" items as well as "perished items" which require repair. The repair requires a positive service time. This is a special class of inventory with two types of commodities in the inventory in which one of them requires negligible service time while the other requires a positive service time.

The second model we propose to investigate is a retrial queueing model in which the units (customers) in the orbit undergoing decay or perishing. Our objective is to control such a model and obtain the optimal utilization policy. We propose to analyze different real life situations with different types of decay rate and perishing rate.

The third model is a variant of a second model. We try to design a model which idle time of the server is decreased by introducing a search mechanism. The search mechanism will go for search of perishable items in the orbit and try to 'promote' the service of perishable items as early as possible. We intend to incorporate the special class of tractable Markov renewal process, namely MAP or Phase type (PH) distribution in this case.

**Mathematical Modelling: A Historical Note**

The word "modeling" comes from the Latin word *modellus*. Mathematical models are abstract representations of realty. Abstract representations of real-world objects have been in use since the stone age. Cavemen paintings revealed that the real breakthrough of modeling came with the cultures of the Ancient Near East and with the Ancient Greek. Use of numbers is documented since about 30,000 BC. Numbers were considered to be the recognizable models. It is well known that by 2,000 BC at least three civilizations Babylon, Egypt, and India had a decent knowledge of mathematical models to improve their every-day life.

The development of philosophy in the Hellenic Age and its connection to mathematics lead to the deductive method.The origin of*mathematical theory*is attributed to the deductive method. As about 600 B.C, Thales of Miletus started using geometry as an effective tool for analyzing reality. Modeling using geometry were further developed by Plato, Aristotle, and Eudoxes at about 300 BC. The summit was reached by Euclid of Alexandria in the same period when he wrote *The Elements*, a collection of books containing most of the mathematical knowledge available at that time. Euclid presented an 'axiomatic' description of geometry in *The Elements* through 'postulates'. This attempt gave rise the first concise axiomatic approach to the mathematical modeling. Around 250 BC Eratosthenes of Cyrene, estimated the distances between Earth and Sun and Earth and Moon and, the *circumference of the Earth* by a geometrical model. Some historians considered Eratosthenes of Cyrene as the first "applied mathematician".Diophantus of Alexandria about 250 AD developed the beginnings of *algebra* based on symbolism and the notion of a *variable*. This was recorded in his book *Arithmetica*.

A land mark mathematical model describing the mechanics of celestial bodies was developed by Ptolemy in 150 AD. The model was so accurate to predict the movement of sun, moon, and the planets.It was used until the time of Johannes Kepler in 1619, when he finally found a superior model for planetary motions. Later Kepler's model was further modified and refined by Issac Newton and Albert Einstein and the resultant model is still valid today.

The modern concept of*algorithm* is credited to the 8th century Arabian mathematicians Abu Abd-Allah ibnMusa. He wrote two famous books. The first was about the Indian numbers—today called Arabic numbers and the second was about the procedures of calculation by adding and balancing His books contain many mathematical models and problem solving algorithms for real-life applications in the areas such as commerce, legacy, surveying, and irrigation. The term *algebra*was taken from the title of his second book.

The probably first great western mathematician who significantly contributed to Mathematical Modelling after the decline of Greek mathematics was *Fibonacci, Leonardo da Pisa* (ca. 1170–ca. 1240). His most influential book is *Liber Abaci*, published in 1202.It began with a presentation of the ten "Indian figures" (0, 1, 2,..., 9), as he called them. He is the man who finally brought the number *zero* to Europe, an abstract model of *nothing*. The book itself was written to be an algebra manual for commercial use.

Artists like the painter Giotto (1267–1336) and the Renaissance architect and sculptor Filippo Brunelleschi (1377–1446) started a new development of geometric principles, especially called perspective drawing. In that time, *visual models* were used as well as mathematical ones. Anatomy is a typical example..

In the later centuries more and more mathematical models were detected, and the complexity of the models increased they address the actual real life situations. It took another 300 years until Cantor and Russell that the true role of variables in the formulation of mathematical theory was fully understood. *Physics* and the description of Nature's principles became the major driving force in modeling and the development of the mathematical theory. Later *economics* joined the group, and now an ever increasing number of applications demand models and their analysis.

The combination of science and modeling leads to a complete understanding of the phenomenon being studied. The uncanny accuracy that Wigner describes extends to all aspects of mathematical modelling and theorizing. Mathematical representations are often based on crude experience, but they are also based on intrinsic limitations regarding what can be mathematically achieved.

Models are considered to be vehicles for learning about the world. Studying a model we can discover features of and ascertain facts about the system the model stands for. So, significant parts of scientific investigation are carried out on models rather than on reality itself. Thus, models allow for surrogative reasoning. For instance, we study the nature of the hydrogen atom, the dynamics of populations, or the behavior of polymers by studying their respective models. This cognitive function of models has been widely acknowledged in the literature, and some even suggest that models give rise to a new style of reasoning, so-called ‘model based reasoning’. Hughes [16] provides a general framework for discussing this question. According to his so-called DDI account, modeling takes place in three stages: denotation, demonstration, and interpretation.

Learning about a model happens at two places, in the construction and the manipulation of the model. Depending on what kind of model we are dealing with, building and manipulating a model amounts to different activities demanding a different methodology. An important class of models is of mathematical nature. In some cases it is possible to derive results or solve equations analytically. But quite often this is not the case. It is at this point where the invention of the computer had a great impact, as it allows us to solve equations which are otherwise intractable by making a computer simulation. Many parts of current research in both the natural and social sciences rely on computer simulations. The formation and development of stars and galaxies, the detailed dynamics of high-energy heavy ion reactions, aspects of the intricate process of the evolution of life as well as the outbreak of wars, the progression of an economy, decision procedures in an organization and moral behavior are explored with computer simulations, to mention only a few examples.

Once we have knowledge about the model, this knowledge has to be ‘translated’ into knowledge about the target system. It is at this point that the representational function of models becomes important again. Models can instruct us about the nature of reality only if we assume that the aspects of the models have counterparts in the world. The author refers to Leng, Mary [19], Morgan [20] and Morrison, Margarat [22] for learning from models.

**Origin and the Relevance of the Research Problem.**

This project discuss the inventory management of perishable items. It also emphasis the importance of appropriately managing the inventory of perishables. The analysis of perishable inventory systems is primarily focused on the tactical question of which inventory control policies to use and the operational questions of how perishables can effectively managed. Insight derived for managing perishable inventory is much more valid for commodities and objects with short life span.

Most products become out date or lose their market value over time. Some products lose value faster than others and they are known as perishable products. Traditionally, perishables outdate due to their chemical structure. Examples of such perishable products are fish products, food products, dairy products, meat, drugs, vitamins etc. But today there are products which outdate because of changes in “market conditions”. Personal computers, computer components such as micro-processors, memory, data storage units, cellular phones, digital cameras, digital music players, smart watches and fashion designer dresses are examples of such products that rapidly lose market value. The life cycles of such products are getting shorter every year due to technological advances. Perishability and outdating are a concern not only for these consumer goods, but for industrial products as well. Recently it was observed by chemical scientists that adhesive materials used for plywood lose strength within 7 days of production.  Blood - one of the most critical resources in health care supply chains is another important example.

Modeling in such an environment implies that at least one or both of the following holds. First, demand for the product may decrease over time as the product ages.This is simply because, the reduced life time will decrease the utility and quality of the product. Second, operational decisions can be made more than once during the lifetime of the product. Either of these factors make the analysis of perishable inventory models a challenge.

**Review of Research and the development in the Perishable Inventory Models**

One of the pioneer papers on Perishable Inventory models was by Derman and Klein[11] in 1958. In this paper, it was assumed that an item which is issued at an age s has a "field life" of L(S) where L is a known function. The general approach was to specify conditions on L for which issuing either the 'First in First Out (FIFO) or the 'Last in First Out'(LIFO) is optimal. Note that in FIFO, the oldest and in LIFO the newest is being issued. There are mainly two streams in such models. They are models with deterministic demand and stochastic demand. We consider the stochastic demand models as they are more common and realistic. But one important thing we need to note here is that, the stochastic perishable inventory models are more complex and hence the analysis is cumbersome. In 1958, Arrow et al [1] considered a Newsboy problem in which the life time is assumed to be exactly one period. Hence the ordering decisions in successive periods are independent. In 1964, Bulinskaya[7] considered a model in which the delivery perishes immediately with probability p and after one period with probabilty1-p. The first perishable inventory model with multiechelon system was considered by Yen [32] in 1965.This pioneer paper pave path to modelswhich consider both allocation and ordering problem.

The first analysis of optimal policies for a fixed life perishable commodity was due to Van Zyl. Later in early 1970's Nahmian and Pierskella [23] improved such models through a series of papers.In late 70's and in early 80's several papers emerged considering set up costs and optimalpolicies are being continuously reviewed.In 1975, Cohen[] finds the critical number S that minimized the expected cost. The first paper considering the analysis of ordering of perishable goods subject to uncertainty in both the demand and the life time was due to Nahmias [23] in 1974.

In 1960, Millard[21] applied the theory of perishable inventory model to manage the stocking of blood. It is interesting to note here that primarily the interests of researchers of perishable inventory models were concentrated on the management of blood banking system during 1970's.  Reasons for this might be the blood bank research was supported by public funds! But gradually food management also came to the picture.

For further review on inventory models, the author refers to Berman et al [6], Cohen [9], Hadley and Whitin [15].

## Methodology

Mathematical representations of physical systems are constructed on the basis of uncertainty. In other words. The real life situations and phenomenon have random nature. So, ideally we refer to these phenomenon as random process or stochastic process. In this project, we concentrate on a stochastic inventory management using the techniques of Queueing Theory.

Results derived for Queueing theory with impatient customers can be used for the analysis of certain type perishable inventory models. This project mainly rely on this technique. Consider a single server queue in which customers will wait for arandom amount of time and leaves the system without service because of impatience. In this case, we can identify the queue with the inventory, the service process with the demand, the time to impatience with the life time of fresh stock, and the arrival of customers to the replenishment of the inventory.

Inventory, Queueing and Reliability are three areas of Applied Probability. They have much in common and can be the same mathematical techniques and procedures. First work on Queueing theory was *The Theory of Probabilities and Telephone conversations* by A.K. Erlang published in 1909.Telephone systems remained the principal application of the queueing theory through about 1950. The trend rapidly changed during the II world war and numerous other applications were found. The techniques of queueing theory can be seen in Cooper [10].

In 1950's a new class of queueing models namely retrial queues were emerged. Retrial queues were also originated with the problems in telephone networking and communication. The standard models of telephone systems, are queueing systems with losses. In the real life situatios, the flow of calls circulating in a telephone network consists of two parts: the flow of primary calls and the flow of repeated calls. The flow of these repeated calls are the consequences of the lack of successes of previous attempts. The standard queueing model do not take into account the flow of repeated calls. Retrial Queues are characterized by the following way. A customer arriving when all servers accessible for him are busy leaves the service area. These unsatisfied customers are viewed as joining a virtual queue called 'orbit'. After some random time they repeatedly make the attempt to reach the server and get the service. One of the earliest papers on Retrial queues was *On the Influence of Repeated Calls in the Theory Of Probabilities of Blocking* by L. Kosten [1947]. For a systematic account of the fundamental methods and results on this topic, we refer to the the monograph by Falin and Templeton[14] and the bibliographical information in Artalejo[2, 3,]. A comprehensive discussion of similarities and differences between retrial queues and their standard cunterparts is given in Artalejo[4]. Comprehensive surveys of retrial queues can be seen in Falin[13] and Yang [31]

The investigation of many of the stochastic processes is essentially very difficult. Except for a few special cases explicit results are very rare. Since the equilibrium distribution of the system state is expressed in terms of contour integrals or as limit of extended continued fraction, they are not convenient for practical applications. More useful is the implementation of the computational probability. By computational probability, we mean the study of stochastic models with a genuine added concern for algorithmic feasibility over a wide, realistic range of parameter values. Hence, numerical investigation to bring out the qualitative behavior of stochastic process is very important.

The progress in computing and communications made in the last quarter of the past century has not only ushered in the "Information Age", but it has also influenced the basic sciences, including mathematics, in fundamental ways. Mathematics can now argument classical techniques of analysis, proof and solution with an algorithmic approach in a manner that enables the consideration of more complex models with wider applicability, and also obtain results with greater practical value to the society. It was strongly supported by the significantly increased computing power.

Among the areas exemplifying all these, a notable one is algorithmic methods for stochastic models based on the "Matrix Geometric Method". Ever since Neuts [24, 25] introduced matrix geometric methods in 1970's interest in this are growing. By the introduction of this method, the "Laplacian Curtain" which covers the solution and hides the structural properties of many interesting stochastic models y lifted. The matrix geometric methods comes under broader heading of computational probability. A wide variety of stochastic models, the steady-state and occasionally the transient measures of the underlying process can be expressed in terms of matrix R or G. The G matrix is a modified version R matrix in the matrix geometric method. The new version namely, "matrix analytic method" was introduced by Ramaswami [27]. These matrices are the minimal non-negative solutions to a non-linear equation.

Analysis of a realistic and practical perishable inventory model is difficult and closed form solution is almost impossible. Our subject matter is attempting the analytical and algorithmic solution of a Stochastic process a perishable inventory model. Our aim is to set the tools that go by the name "matrix geometric methods' if the analytic closed form solution is not possible. We develop a FORTRAN code for the performance analysis of such models.

CHAPTER 2

# An Inventory Model with Perishable Items Require Positive Service Time

In this chapter, we analyze an inventory model in which there are demands for “new” items as well as “perished items” which require repair. We could see lots of real life examples in the market of readymade dress materials, ornaments, and vehicles- For example, consider a vehicle dealer who sells both brand new vehicles as well as used vehicles. Used vehicles are considered as perished items. The service time for these demands may be different. If a demand occurs for a brand new item, the dealer can fulfill the demand without any time delay. But in the case of a perished item the dealer requires a non-zero service time for processing the item. Krishnamoorthy et al [18] considered the control policies for an inventory model with service time.

The model is described as follows. We assume that there are demands either for perished items or those for brand new items. We assume that perished item requires a positive serve time and the brand new item requires negligible service time. Even when a service is going on for processing the perished item, customers may arrive and ask for brand new items. We also assume that the server (the dealer)can serve the customers without interrupting the processing of the perished items. In this model, we get analytical solution. Also we use analytical method to complete the problem.

This chapter is arranged as follows. In section 2 the model is described and is formulated mathematically. In section 3, stability of the system is analyzed and the stability conditions are obtained. In section 4, the steady state distribution of the system is investigated. We obtain a number of performance measures of the system in section 5. The performance measures help us to the system. In the last section, cost analysis of the model is performed. We make use of the performance measures and assign suitable cost to them for cost analysis. We construct a cost function which is a function of the probability that a customer demands processed items, then it is seen to be a convex function of this probability and hence has global minimum. It is proved that irrespective of other costs involved, this function is minimum when the probability of demanding processed item is very small. This section provides expected cost of running the system. The cost analysis help us to control the system optimally. In particular, we prove that the optimal reorder level is zero. Numerical illustration are also provided in that section.

## 2.1 The Model description and Mathematical formulation

We consider an (*s, S*) inventory system positive service time for processing demands for perishable items. Demands are assumed to arrive according to a Poisson process of rate λ. Out of these arrivals, a fraction $\lambda_1 = \lambda_p, 0<p<1$ are for items requiring processing and the rest

$\lambda_2= \lambda(1-\beta)$ are for brand new items which does not require any processing. The service time of demands for brand new items is negligible. This is exactly as in the case of classical inventory models, see for example Hadley and Whitin [15]). In the latter case we say that service rate is infinity. This type of inventory systems are very common.

A more concrete example is the following: Consider the transaction of electronic goods. A few demands will be for assembled (hence requiring service time) items (such as a computer) where as the rest of the demands arise for the assembled one.

In some cases, the processing cost may be heavy. Hence, the dealer prefers catering demands for brand new items (that is, the items requiring no processing). In inventory, customer satisfaction and goodwill are prime objectives. So, the dealer has to entertain demands for processed items also. If lead time is zero and no shortage is permitted, then a queue of customers who need processed item will alone be formed. Demands for other type will be instantly satisfied and hence no queue of such customers get generated. While the server is engaged in serving a customer requiring processed items, the demand for the item which requires no processing can be simultaneously satisfied without any interruption to the ongoing service. Demand processing starts only when that customer is in the head of the queue.

We assume that time of processing follows exponential distribution with parameter $\mu$. We introduce the following notations.
$N(t)$= Number of customers in the system at time $t$

$I(t)$ = Inventory level at time $t$,

which ranges over $\{s, s+1,\ldots S\}$.

Then $\{(N(t), I(t)),\ t\ \varepsilon R\}$ is a continuous time Markov chain on the set $\{(o, j)\,|\,s \le j \le S-1\}\cup\{(i, j)\,|\,i\ge 1; s+1\le j \le S\}$ which is clearly a level independent quasi-birth-death process(LIQBD). Note that if at a service completion epoch, no other customer is left in the system, then if the inventory level falls to $s$, under
these setup the infinitesimal generator $Q$ of the Markov chain is

$$Q=\begin{pmatrix} B & A & 0 & 0 & \ldots \\ A_2 & A_1 & A & 0 & \ldots \\ 0 & A_2 & A_1 & A & \ldots \\ \vdots & & \vdots & & \end{pmatrix}$$

This is a quasi-Toeplits matrix. The entries are described as follows. Denote by i̲ the set of elements in the state space $\{(i\ ,\ s+1),(i,\ s+2),\ldots,(i,\ S)\}$ for $i \geq 1$ and by 0̲ the set $\{(0,\ s),\ (0,\ s+1,\ldots,(0,\ S-1))\}$. We call the first component in a pair, the level and the second coordinate is referred to as phase. Note that when a demand for unprocessed item arises there is a transition within a level and when the service of a customer demanding processed item is completed there is a decrease in level by one unit (the served customer leave the system). Similarly due to arrival of a customer to the system who requires processed item, the level increases by unit. Thus the LIQBD process under consideration is skip-free to the left as well as to right.

Now we describe the entries in $Q$:

$$\mathbf{B} = \begin{pmatrix} -\lambda & 0 & 0 & \ldots & 0 & \lambda(1-p) \\ \lambda(1-p) & -\lambda & 0 & \ldots & 0 & 0 \\ 0 & \lambda(1-p) & -\lambda & & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & & -\lambda & 0 \\ 0 & 0 & 0 & \ldots & \lambda(1-p) & -\lambda \end{pmatrix}$$

Describes transition within zero level;

$$A = \begin{pmatrix} \lambda p & 0 & 0 & \ldots & 0 & 0 \\ 0 & \lambda p & 0 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & \lambda p & 0 \\ 0 & 0 & 0 & \ldots & 0 & \lambda p \end{pmatrix}$$

Describe transition from level $i-1$ to level $i$, $i \geq 1$;

$$A_1 = \begin{pmatrix} -(\lambda+\mu) & 0 & 0 & & 0 & \lambda(1-p) \\ 0 & \lambda(1-p) & -(\lambda+\mu) & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & -(\lambda+\mu) & 0 \\ 0 & 0 & 0 & \ldots & \lambda(1-p) & -(\lambda+\mu) \end{pmatrix}$$

Which describes transitions with level $i$, $i \geq 1$; and

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & \dots & 0 & \mu \\ \mu & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & \mu & 0 & \dots & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & \mu & 0 \end{pmatrix}$$

Stands for transition from level $i$ to $i-1$, $i \geq 1$.

All these are square matrices of order $S - s$. note that these matrices have nice structure. It is these structures, especially that of $A_2$ that prompted us to conjecture that the system is analytically solvable which in turn took to a matrix E defined as

$$E = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

**2.2 Stability Condition and Steady State Probability**

In this section, we examine the stability of the system. Let A be the matrix defined by

$A = A \;\; + A_1 + A_2 =$

$$\begin{pmatrix} -((1-p)\lambda+\mu) & 0 & \cdots & 0 & \lambda(1-p)+\mu \\ \lambda(1-p)+\mu & -((1-p)\lambda+\mu) & \cdots & \vdots & 0 \\ 0 & \lambda(1-p)+\mu & \cdots & \vdots & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & & & \lambda(1-p)+\mu & -((1-p)\lambda+\mu) \end{pmatrix}$$

Let $\pi = (\pi_1, \pi_2, \ldots \pi_{S-s})$be the stationary probability vector associated with the generator A, ie., $\pi A = 0$. Then we immediately get $\pi_1 = \pi_2 = \ldots = \pi_{S-s} = 1/Q$ where $Q = S-s$, the order quantity at each order placement epoch. Then we have the system stability condition given by the relation $\pi A\ e < \pi A_2 e$ which results in $\lambda p < \mu$ where $= (1\ldots\ldots.1)'$.

Thus we have

THEOREM 3.1. The system is stable if and only if $\lambda p < \mu$.

Note that $\lambda_2 = \lambda(1-p)$ does not play any role in the stability condition. This is attributed to the fact that the fraction $\lambda_2$ of customers are instantly served (ie ,. Their rate of service is infinity) as they need only unprocessed item requiring negligible service time. Thus the theorem says that even λ is very large steady state distribution the system can be stable provided p is corresponding small and $\lambda p < \mu$.

Now let $X = (X\ , X_1, X_2 \ldots.)$ be the stationary probability vector of the LIQBD process Here vectors Xi s' are of order S-s. Then $XQ = 0$ and $Xe' = 1$ where $\underline{e} = (1, 1, \ldots)$ whose components are all 1s'. the above relation leads to :

$X\ B\ + X_1 A_2 = 0$

$X_{i-1} A\ + X_i + A_1 + X_{i+1} + A_2 = 0$ for $i \geq 1$

The first of these gives $X_1 = X\ (-B\ ) A_2^{-1}$.

Since in the classical (s,S) inventory models with renewal/compound renewal demands, where lead time is zero and shortage cost is infinity the long run inventory level distribution is uniform, we try a solution of the form $X\ = \gamma(1,1,\ldots.. 1)$. Then $X_1 = \gamma(1,1,\ldots.. 1).(-B\ )1/\mu E$, where γ is a constant to be determined. The above relation, on simplification, gives $X_1 = (\lambda_p/\mu) X_0$
On simplification, the steady state probability $X_n = (1-\lambda_p/\mu)^n\ (1/Q,,1/Q,,\ldots..1/Q)$.
The above expression provides a product form solution.

## CHAPTER 3

## A Retrial Queue with Perishing of Customers in the Orbit

In this chapter, we investigate a retrial queueing model in which the units (customers) in the orbit undergoing decay or perishing. The model is motivated by the following interesting examples.

Consider a car servicing station where car owners are registered for usual service. The registered customers are considered to be in a virtual queue called 'orbit'. If the server is free at an arrival the customer directly goes to the server and the service begins. If the customer finds the server busy at an arrival, it goes to the virtual queue called orbit. From the orbit, the customer is again trying for service. Immediately after each service a competition takes place between the primary (new) customers and the secondary (orbit) customers. In the meantime, the customers waiting in the orbit undergoing some sort of 'perishing'. Such customers requires immediate service or in fact the repair. In such situations, immediately after each service, those customers are taken for service.

Another important example is people waiting in the 'orbit 'for usual medical checkup. There may arise situation where the customers waiting in the orbit require immediate medical attention. This is exactly the same case as in the previous example.

### 3.1 Model 1: General Case

We consider a single retrial queueing system, in which customers are arriving according to a Poisson process with rate $\lambda$. If an arriving customer finds the server free, it directly enters into service. The service rate is assumed to follow exponential distribution with rate $\nu$. On the other hand, if the arriving customer finds the server busy, it goes to the virtual waiting line called the orbit. The customers in the orbit are also repeating their attempts to get into the server. The retrial process is assumed to follow Poisson distribution with $\mu$. This is the typical retrial queueing system. In the men time, the customers in the orbit are undergoing 'perishing' and turns into a situation when it requires immediate service. The perishing process is assumed to follow Poisson distribution with rate $\theta$. We consider the most general case in which the perishing and the retrial rate depends on the number of customers present in the orbit. So, the flow from the orbit to the server is $\alpha+n(\mu+\theta)$, where n is the number of customers present in the orbit.It is quite realistic to assume that he retrial rate and the perishing ratedepends on the numbers of units in the orbit. Though these assumptions make the mathematical formulations cumbersome and analysis complex, we get the closed solutions in terms of Hyper Geometric Functions. The classical retrial set up, each service is preceded and followed by an idle period. This idle period is terminated by the arrivalfrom the orbit or by the primary customer. But in our problem, immediately after the service, if there is a perished customer in the orbit, the server takes the perished one for service. In our model, after the completion of each service, a competition takes place between primary arrival, secondary arrival from the orbit and perished customer from the orbit.

**Mathematical Formulation and Steady State Analysis of the Model**

Let N(t) denote the number of customers in the orbit, and C(t) = 0, if the server is free and C(t) = 1 if the server is busy. Then N(t) = 0,1,2,……..The (C(t),N(t)) forms a continuous time Markov chain.

Set of statistical equilibrium equations are

$[\lambda + \alpha(1 \quad \delta_{on}) + n(\mu+\theta)]P_{0n} = \nu P_{1n}$ (1)

$(\lambda+\nu) P_{1n} = \lambda P_{0n} + \lambda P_{1,n-1} + [\alpha(n+1)(\mu+\theta)]P_{1,n+1}$ (2)

Using (1) eliminate probabilities of $P_{1n}$ from (2) and then it reduces to

$\nu [\alpha+n(\mu+\theta)] P_{0n} - \lambda[\lambda+ \alpha+(n-1)(\mu+\theta)] P_{0n}$

$(\lambda+\nu) P_{1n} = \lambda P_{0n} + \lambda P_{0,n-1} + [\alpha+(n+1)(\mu+\theta)] P_{0,n-1}$ (3)

This implies that

$\nu [\alpha+n(\mu+\theta)] P_{0n} - \lambda[\lambda+ \alpha+(n-1)(\mu+\theta)] P_{0,n-1}$

ie, $P_{0n} = \frac{\lambda[\alpha+(n-1)(\mu+\theta)]}{\nu [\alpha+n(\mu+\theta)]} P_{0,n-1}$

$\rho^n \prod_{j=0}^{n-1}[\frac{\lambda+\alpha(1-s_{oj})+j(\mu+\theta)}{\alpha+(j+1)(\mu+\theta)}] P_{00}$ (4)

Now from (1), and (4)we obtain

$P_{1n} = \rho^{n+1} \prod_{j=1}^{n}[\frac{\lambda+\alpha+j(\mu+\theta)}{\alpha+j(\mu+\theta)}] P_{00}$ (5)

Applying the normalizing condition $\sum_{n=0}^{\infty} P_{0n} + \sum_{n=0}^{\infty} P_{1n} = 1$

We get, $(P_{00})^{-1}$

$= 1+ \frac{\lambda\rho}{\alpha+\mu+\theta}$ 2F1 $(1, \frac{\lambda+\alpha}{\mu+\theta}) + 1 ; \frac{\alpha}{\mu+\theta} +2; s)$

$+s$ 2F1 $(1, , \frac{\lambda+\alpha}{\mu+\theta} +1 ; \frac{\alpha}{\mu+\theta} + 2 ; s)$ (6)

Partial generating function $P_0(z) = \sum_{n=0}^{\infty} z^n P_{0n}$ and

$P_1(z) = \sum_{n=0}^{\infty} z^n P_{1n}$ are given by

$P_0$ (ʒ)= $P_{00}$ {1+ $\frac{\lambda\rho ʒ}{\alpha+\mu+\theta}$ 2F1 (1, $\frac{\lambda+\alpha}{\mu+\theta}$) + 1 ; $\frac{\alpha}{\mu+\theta}$ +2; sʒ ) } (7)

And $P_{01}$(ʒ)= s$P_{00}$ $2F1${1, $\frac{\lambda+\alpha}{\alpha+\mu+\theta}$ 2F1 (1, $\frac{\lambda+\alpha}{\mu+\theta}$ + 1 ; $\frac{\alpha}{\mu+\theta}$ +2; sʒ ) } (8)

If N(t) denote the number of customers in the orbit then the generating function of its stationary distribution is given by P (ʒ) = $P_0$(ʒ) +$P_1$(ʒ)

In particular, the first factorial moment of the number of customers in the orbit is

Type equation here.

$\phi_1$ = E[N(t)] = $\frac{A}{B}$, where (9)

A = λ$s$ 2F1(1, $\frac{\lambda+\alpha}{\mu+\theta} + 1$ ; $\frac{\alpha}{\mu+\theta} + 2; s$ )

+$(\lambda + \alpha + \mu + \theta)$ $s^2$ 2F1 (2, , $\frac{\lambda+\alpha}{\mu+\theta} + 2; \frac{\alpha}{\mu+\theta}; s$)

+$\frac{\lambda(\lambda+\alpha+\mu+\theta)s^2}{\alpha+2(\mu+\theta)}$ 2F1 (2, $\frac{\lambda+\alpha}{\mu+\theta} + 2; \frac{\alpha}{\mu+\theta} + 3; s$

And B = $(\alpha + \mu + \theta)$ + $(\alpha + \mu + \theta)$s 2F1 (1, $\frac{\lambda+\alpha}{\mu+\theta} + 1; \frac{\alpha}{\mu+\theta} + 1; s$ )

+ λs 2F1 $(1, \frac{\lambda+\alpha}{\mu+\theta} + 1; \frac{\alpha}{\mu+\theta} + 2; s$ )

Expected cycle length = E(T) = $\frac{1}{\lambda P00}$

Expected idle time in a cycle E(Id) = $\sum_{n=1}^{\infty} \frac{p_{0n}}{p_{00}}, \frac{1}{\lambda+\alpha+(\mu+\theta)}$

(Here the server is considered to be consisting of two mechanisms one for service and the other for search. We consider the idleType equation here. time for service mechanism only.)

= $\sum_{n=1}^{\infty} s^2 \prod_{j=0}^{n-1} [\frac{\lambda+\alpha+1-s_{0j}+j(\mu+\theta)}{\alpha+(j+1)(\mu+\theta)}] \frac{1}{\lambda+\alpha+n(\mu+\theta)}$

$\frac{\lambda}{(\lambda+\alpha)^2}$\{ 3F2(1, , $\frac{\lambda+\alpha}{\mu+\theta} + \frac{\lambda+\alpha}{\mu+\theta} + \frac{\alpha}{\mu+\theta} + 1 \frac{\lambda+\alpha}{\mu+\theta} + 1; s$)-1}

### 3.2 Model 2 ( A Special Case)

To occupy the server when it is free:

a) Due to a primary arrival with rate λ
b) Due to an orbital arrival with rate of nμ; (when n customers are in the orbit).
c) Due to an arrival from the orbit by search procedure with rate nθ; (here we assume that the server is aware of the status of the orbit number that search take place only when the orbit is non-empty).

$P_{00}(t+\quad t)$ =$P_{00}(t)\quad \lambda P_{00}(t)$ t+$\nu P_{10}(t)\quad t$

$P'_{00}(t)$ = - $\lambda P_{00}(t)\quad t + \nu P_{10}(t)\quad t$

$P_{0n}(t+\quad t)$ =$P_{0n}(t)\quad (\lambda+\mathrm{n}\mu+\mathrm{n}\theta)P_{0n}(t)\quad t+\ \nu P_{1n}(t)$

$P'_{0n}(t)$ = $\quad(\lambda+\mathrm{n}\mu+\mathrm{n}\theta)P_{0n}(t)+\ \nu P_{1n}(t)$

$P_{10}(t+\quad t)$ = $P_{0n}(t)\quad (\lambda+\nu)P_{10}(t)\quad t+\ \nu P_{1n}(t)\quad t+\ \lambda P_{00}(t)\quad ()t$ +(μ + θ) $\nu P_{01}(t)$ t

$P'_{10}(t) = \quad (\lambda+\nu)P_{10}(t)+\ \lambda P_{00}(t)$ +(μ + θ)$P_{01}(t)$

$P_{1n}(t+\quad t)$ =$P_{1n}(t)\quad (\lambda+\nu)P_{1n}(t)\quad t+\ \lambda P_{1,n-1}(t)\quad t+\ \lambda P_{0n}(t)\quad (t)+(n+1)(\mu+\theta)P_{0,n+1}(t$

$P'_{1n}(t) = \quad (\lambda+\nu)P_{1n}(t)+\ \lambda P_{1,n-1}(t)+\lambda P_{0n}(t)+(n+1)(\ \mu+\theta)P_{0,n+1}(t)$

Set of statistical equilibrium equations are

0 =- $\lambda P_{00}\ +\ \nu P_{10}$ .................. (1a)

0 = $\quad(\lambda+\mathrm{n}\mu+\mathrm{n}\theta)P_{0n}+\ \nu P_{1n}$ n ≥ 1 .................... (1b)

0 = -( $\lambda+\nu)P_{10}$ + $\lambda P_{00}\ +(\mu+\theta)P_{01}$ ..................... (2a)

0 = -( $\lambda+\nu)P_{1n}$ + $\lambda P_{1,n-1}\ +\lambda P_{0n}\ (n+1)(\ \mu+\theta)P_{0,n+1}$ Type equation here. n ≥ 1 .....................(2b)

Thus, $P_{1n}$ = $\frac{(\lambda+\mathrm{n}\mu+\mathrm{n}\theta)}{\nu}P_{0n}$ Type equation here.n ≥ 1 .................... (3)

∴ $P_{1,n-1}$ = $\frac{(\lambda+(\mathrm{n}-1)+(\mu+\theta)}{\nu}P_{0n-1}$ n ≥ 2 ................... (4)

Solving the above equation, we get

For all n ≥ 2, $\mathrm{n}\,(\mu+\theta)\,\nu P_{0n} - \lambda[\lambda + (\mathrm{n} \quad 1)(\mu+\theta)]P_{0n-1} = 0$ ........(5)

$\nu\,(\mu+\theta)P_{01} \quad \lambda^2 P_{00} = 0$ ....................(6)

$$P_{01} = \frac{\lambda^2}{\nu\,(\mu+\theta)}P_{00} = \rho\frac{\lambda}{(\mu+\theta)}P_{00} \qquad \text{.................... (7)}$$

And $\quad P_{0n} = \frac{\lambda[\lambda+(\mathrm{n}-1)+(\mu+\theta)]}{\nu\,\mathrm{n}(\mu+\theta)}P_{0n-1}$

$= \rho\frac{\lambda+(\mathrm{n}-1)+(\mu+\theta)}{n\,(\mu+\theta)}P_{0n-1} \qquad \mathrm{n} \geq 2$ ..................... (8)

Solving, are get

$P_{0n} = \frac{\rho^n}{n!(\mu+\theta)^n}\prod_{i=1}^{n}[\lambda + (\mathrm{n} \quad 1) + (\mu+\theta)]P_{00} \qquad \mathrm{n} \geq 0 \qquad (9)$

$P_{1n} = \frac{\rho^{n+1}}{n!(\mu+\theta)^n}\prod_{i=1}^{n}\lambda + \mathrm{i}(\mu+\theta)\,P_{00} \qquad \mathrm{n} \geq 0 \qquad (10)$

Applying the normalizing condition

$\sum_{n=0}^{\infty}P_{0n} + \sum_{n=0}^{\infty}P_{1n} = 1$

$$\left\{\sum_{n=0}^{\infty}\frac{\rho^n}{n!(\mu+\theta)^n}\prod_{i=1}^{n}\lambda + (\mathrm{n} \quad 1) + (\mu+\theta) + \sum_{n=0}^{\infty}\frac{\rho^{n+1}}{n!(\mu+\theta)^n}\prod_{i=1}^{n}\lambda \ + \mathrm{i}(\mu+\theta)\,P_{00} = 1\right\} \qquad (11)$$

Now $\sum_{n=0}^{\infty}\frac{\rho^n}{n!(\mu+\theta)^n}\prod_{i=1}^{n}\lambda + (\mathrm{n} \quad 1) + (\mu+\theta)$

$= \ (1 \quad \rho)^{\frac{-\lambda}{1+\theta}} \qquad (12)$

Now, $\sum_{n=0}^{\infty}\frac{\rho^{n+1}}{n!(\mu+\theta)^n}\prod_{i=1}^{n}[\lambda \ + \mathrm{i}(\mu+\theta)]$

$= \rho\,(1 \quad \rho)\mu^{\frac{-\lambda}{+\theta}-1} \qquad (13)$

Thus $P_{00}{}^{-1} = (1 \quad \rho)\mu^{\frac{-\lambda}{+\theta}-1}\{(1 \quad \rho) + \rho\}$

Ie, $P_{00}{}^{-1} = (1 \quad \rho)\mu^{\frac{-\lambda}{+\theta}-1}\{(1 \quad \rho) + \rho\}$

Ie, $P_{00}{}^{-1} = (1 \quad \rho)\mu^{\frac{-\lambda}{+\theta}-1}$

Ie, $P_{00}{}^{-1}$ = $(1 \quad \rho)\mu^{\frac{-\lambda}{+\theta}+1}$ (14)

The corresponding partial generating functions are :

$p_0(\mathfrak{z})$ = $\sum_{n=0}^{\infty} p_{0n}\, \mathfrak{z}^n$

= $\sum_{n=0}^{\infty} \frac{s^n}{n!(\mu+\theta)^n} \prod_{i=1}^{n} [\lambda \;\; + (i \quad 1)(\mu+\theta)] p_{oo}\, \mathfrak{z}^n$

Ie, $p_0(\mathfrak{z})$ = (1-ρ) $[\frac{1-\rho}{1-\rho\mathfrak{z}}]^{\frac{\lambda}{\mu+\theta}}$ (15)

$p_1(\mathfrak{z})$ = $\sum_{n=0}^{\infty} p_{1n}\, \mathfrak{z}^n$

= $\sum_{n=0}^{\infty} \frac{\rho^{n+1}}{n!(\mu+\theta)^n} \prod_{i=1}^{n} [\lambda \;\; + i(\mu+\theta)]\, p_{oo}$

$p_1(\mathfrak{z})$ = ρ $[\frac{1-\rho}{1-\rho\mathfrak{z}}]^{\frac{\lambda}{\mu+\theta}} + 1$ (16)

the stationary distribution of the number of sources of repeated calls $q_n$ = P[n(t) = n]

p($\mathfrak{z}$) =$p_0(\mathfrak{z})$ +$p_1(\mathfrak{z})$

=(1+ρ -ρ$\mathfrak{z}$) $\{\frac{1-\rho}{1-\rho\mathfrak{z}}\}^{\frac{\lambda}{\mu+\theta}+1}$ (17)

Thus fractional moments $\phi_n$ = E[N(t)]n of the queue lengths are given by

Ie, $\phi_n$ = p(1)

In particular, $\phi_1$= p(1)

Ie , E[N(t)] = $\phi_1$

= -$\{\frac{\rho(\lambda+(\mu+\theta))}{(\mu+\theta))(-1+\rho)}\}$

$\phi_2$ = p$^{(2)}$(1) = $\frac{(\theta+c+\mu)\rho^2(\lambda+2(\theta+\mu)\rho)}{(\theta+\mu)^2(-1+\rho)^2}$

Var (N(t)) =$\phi_2 \quad \phi_1 - \phi_1{}^{2}$

=$\frac{\rho(\lambda-(\mu+\theta)\rho(-1-\rho+\rho^2))}{(\mu+\theta)(-1+\rho^2)}$

The stationary distribution of the number of the customers is the system Qn =P[K(t)] = n has the generating function

Q($\mathfrak{z}$) = $P_o$ ($\mathfrak{z}$) + $\mathfrak{z}P_1(\mathfrak{z})$

= $[\frac{1-\rho}{1-\rho\mathfrak{z}}]^{\frac{\lambda}{\mu+\theta}+1}$

$\varphi_1$= $Q^{(1)}$ (1) = $1+\frac{\frac{\lambda}{\mu+\theta}}{1-\rho}\rho$

Thus E[K(t)] = $1+\frac{\frac{\lambda}{\mu+\theta}}{1-\rho}\rho$

Thus var (K(t)) = $\varphi_2 + \varphi_1 \quad \varphi_1{}^2$

$$= \frac{(\theta+\lambda+\mu)^{\rho}}{(\theta+\mu) \quad (-1+\rho)^2}$$

The blocking probability $p_1$ = $p_{1(1)}$ = ρ

### 3.3 Particular Case of 3.1 and 3.2

In this section we consider the special cases of 3.1 and 3.2 in which the retrial rate and the perishing rate are independent of the number of units present in the orbit. Assumptions are exactly the same as those in sections 3.1 and 3.2

In this case, the set of statistical equilibrium equations are

$\lambda P \quad = \nu P_1$ .....................(1. a)

$(\lambda + \eta\mu + \eta\theta) \quad P_{0n} = n\nu P_{1n}$ ......................(1.b)

$(\lambda + \nu)P_1 \quad = \lambda P_1 \quad + (\mu + \theta)\, P_{\ 1}$ .......................(2.a)

$(\lambda + \nu)P_{1n} = \lambda\, P_{1,n-1} + \lambda P_{0n} +(n+1)(\mu+\theta)P_{0,n+1}$ ........................(2.b)

Thus,

$(n+1)(\mu+\theta)\nu P_{1,n+1} - \lambda(\lambda + n(\mu+\theta))P_{0n} = n(\mu+\theta)\nu\, P_{0,n} - \lambda(\lambda + n(\mu+\theta))P_{0,n-1}$

$P_{\ 1} = \rho\frac{\lambda}{\mu+\theta}P$

$P_{0n} = \rho\, (\frac{\lambda+(n-1)(\mu+\theta)}{n(\mu+\theta)})P_{0,n-1}$

We get

$P_{0n}$= $\frac{\rho^n}{n!(\mu+\theta)^n}\prod_{i=1}^{n}(\lambda+(i\quad 1)(\mu+\theta))P$

and

$P_{1n}$= $\frac{\rho^{n+1}}{n!(\mu+\theta)^n}\prod_{i=1}^{n}(\lambda+i(\mu+\theta))P$

Applying the normalizing condition $\sum_{n=0}^{\infty} P0n$ + $\sum_{n=0}^{\infty} P1n$ =1

We get

P = $(1 \quad \rho)^{\frac{1+\lambda}{\mu+\theta}}$

We consider the Partial generating functions

P (z) = $\sum_{n=0}^{\infty} P0n\, Z^n$

=(1- $\rho$)$[\frac{((1-\rho))}{((1-\rho z))}]\frac{\lambda}{\mu+\theta}$

and

$P_1(Z)$ = $\sum_{n=0}^{\infty} P1n\, Z^n$

= $\rho\left[\frac{((1-\rho))}{((1-\rho z))}\right]1+\frac{\lambda}{\mu+\theta}$

The stationary distribution of the sources of repeated calls $q_n =$ P[N(t)=n] has the generating function P(z) = P (z)+ $P_1(z)$

=(1 + $\rho \quad \rho z)\left[\frac{((1-\rho))}{((1-\rho z))}\right]1+\frac{\lambda}{\mu+\theta}$

The factorial moments $\phi_n = E[N(t)]n$ of the queue length are given by

$$\phi_n = p^{(n)}(1)$$

In particular $\phi_1 = E[N(t)]$ = $\frac{\rho[\lambda+(\theta+\mu)]}{(\theta+\mu)(1-\rho)}$

$$\phi_2 = \frac{(\theta+\lambda+\mu)\rho^2[\lambda+2(\theta+\mu)\rho]}{(\theta+\mu)(1-\rho)^2}$$

Var [N(t)] = $\phi_2 + \phi_1 \quad \phi_1{}^2$

$$= \frac{\rho[\lambda+(\theta+\mu)\rho(1+\rho-\rho^2)]}{(\theta+\mu)(1-\rho)^2}$$

The stationary distribution of the number of customers in the system $Q_n$ = P[K(t)=n] has generating function Q(z) = P (z) + z $P_1$(z)

$$= \left[\frac{(1-\rho)}{(1-\rho z)}\right] 1 + \frac{\lambda}{\theta+\mu}$$

Its factorial moments $\Psi_n =$ E[K(t)]n of the queue length are given by

$$\Psi_n = Q^{(n)}(1)$$

In particular $\Psi_{1=}$ E[K(t)] $= \frac{(\theta+\lambda+\mu)\rho}{(\theta+\mu)(1-\rho)}$

$$\Psi_2 = \frac{(\theta+\lambda+\mu)(2\theta+\lambda+2\mu)\rho^2}{(\theta+\mu)^2(1-\rho)^2}$$

Var [N(t)] = $\Psi_2 + \Psi_1 \quad \Psi_1{}^2 = \frac{(\theta+\lambda+\mu)\rho}{(\theta+\mu)(1-\rho)^2}$

CHAPTER 4

# A Retrial Queueing Model with Server Searching for Perishable Orbital Customers

## Introduction

In this chapter, we consider a retrial queueing model in which the customers in the orbit are perishable. So, the server himself search for customers in the orbit and thereby optimize the service and reduce the loss of customers from the orbit. These types of models are very common in communication networks. The signals waiting in the orbit for service may lose its relevance as time goes on. Sometimes long waiting may weaken the strength of the signal. In these situations, the ideal model is which reduce the idle time of the server. This can be achieved by the 'search ' mechanism introduced by Artelejo et al [5 ] in the retrial queueing context. A concept of search was introduced in the classical queueing set up by Neutes and Ramalhote [26]in 1984. The investigator himself with others [8, 12, and 17] extended the possibilities of the search mechanism to more general models. Immediately after the service completion of each service the server himself goes for search of perishable orbital customers with some assigned probability. The design and control of such real life models with perishable orbital customers can be effectively done by estimating of this probability.

## The Mathematical Formulation and Steady State Analysis of the Model

In this chapter, we consider a retrial queueing system in which perishable customers are arriving according to a Poisson process with parameter $\lambda$. If the customer finds the server busy, it goes to the orbit and repeats the attempts to get the service. The retrial process is also assumed to follow the Poisson process with parameter $\mu$. The service is assumed to follow the exponential distribution with parameter $\nu$. This is the classical retrial queueing set up in which each service is followed by an idle time. Though the customers in the orbit are bound to perish and eager to get service as soon as possible, they may fail to reach the server even when the server is idle. This is simply because of the 'ignorance' of the customer about the state of the server. This is a challenge in the real life models which reduce the efficiency of the model. We use the search mechanism introduced by Artalejo, Joshua and Krishnamoorthy[5] to overcome the above challenge. Upon completion of a service, the server search for customers in the orbit with a probability p. The search process is assumed to follow exponential distribution with parameter β. If a customer is found, a service is initiated. The server remains idle with probability 1-p. In this case, new service will be initiated only a primary arrival or by a retrial customer from the orbit.

Let C(t) be the status of the server.

That is, C(t) = 0 ; if the server is idle

1; if the server is busy by searching

2; if the server is busy by serving.

Let N(t) denote the number of customers in the orbit.

That is, N(t) =0,1,2,…………….

(C(t), N(t)) describes a continuous time Markov chain.

The infinitesimal generator Q of the Markov chain has the form

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 \ldots\ldots \\ B_{10} & A_{11} & A_0 & 0 & 0 & 0 \ldots\ldots \\ 0 & A_{22} & A_{12} & A_0 & 0 & 0 \ldots\ldots \\ 0 & 0 & A_{23} & A_{13} & A_0 & 0 \ldots\ldots \\ 0 & 0 & 0 & A_{24} & A_{14} & A_0 \ldots\ldots \\ & \ldots\ldots & & & & \end{pmatrix}$$

Where $B_{00} = \begin{bmatrix} \lambda & \lambda \\ \nu & (\lambda+\nu) \end{bmatrix}$ ; $B_{01} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

$$B_{10} = \begin{pmatrix} 0 & \mu \\ 0 & \beta \\ 0 & 0 \end{pmatrix}; \qquad A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$A_{1l} = \begin{pmatrix} (\lambda + l\mu) & 0 & \lambda \\ 0 & (\lambda + \beta) & 0 \\ \nu(1 \quad p) & \nu p & (\lambda + \nu) \end{pmatrix}; \ l = 1,2 \ldots\ldots$$

$$A_{2l} = \begin{pmatrix} 0 & 0 & l\mu \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}; \quad l = 2,3\ldots\ldots\ldots$$

Let $x_{ij}$ denote the steady state probability of the state at an arbitrary time and let x be a raw vector with elements $x_{ij}$ .When the queue is stable, x is the unique solution to xQ =0 and xe= 1 where e is the column vector with all elements = 1

The system of equation given by xQ = 0 and xe =1 can be truncated at a sufficiently large value of I, say M and the resulting finite system can be solved for the equilibrium probability vector. M is chosen in such a way that the loss of probability mass due truncation is small. (no analytical basis for the choice of M is available and a trail and error method id adopted).

**Algorithmic solution :-**

For small values of j, the likelihood of an idle server and therefore the likelihood of a retrial request being successful is not small. As j increases, the probability of a successful retrial request progressively decreases. When j is sufficiently large, a majority of the retrial repeats fail to find free server and do not result in a large of state. Further increase in j mainly adds to the number of unsuccessful retrial request. Under their condition , if the number of customers who can generates retrial repeats is restricted to an appropriately chosen number N, the effect on system dynamics and the equilibrium probability vector is minimal. Except when the retrial rate is extremely small, we can expect N to be substantially smaller than the value of M required by the direct truncation to achieve the same degree of accuracy.

The approximation modifies the matrix Q to the following form.

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots & & & & & & & & & \\ B_{10} & A_{11} & A_0 & 0 & 0 & 0 & \cdots & & & & & & & & & \\ 0 & A_{22} & A_{12} & A_0 & 0 & 0 & \cdots & & & & & & & & & \\ \cdots & & & & & & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & A_{2,N-1} & A_{1,N-1} & A_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & A_2 & A_1 & A_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & A_2 & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ \cdots & & & & & & & & & & & & & & & \end{pmatrix}$$

Where $A_1 = A_{1N}$ and $A_2 = A_{2N}$

Let $x = [\, x_0 \;\; x_1 \;\; x_2 \;\; \ldots\ldots\ldots \,]$

Such that $x_0 = [\, x_{00} \;\; x_{20} \,]$ ; $x_1 = [\, x_{01} \;\; x_{11} \;\; x_{21} \,]$ ; $x_2 = [\, x_{02} \;\; x_{12} \;\; x_{22} \,]$ ………………

Suppose that there exist a constant matrix R such that

$x_{N+k-1} = x_{N-1}\, \mathrm{R}^2$ ; K≥1 (1)

The matrix R is the unique non- negative solution with spectral radius lunation.

less than 1 of the equation.

$\mathrm{R}^2\; A_2 + \mathrm{R}\, A_1 + A_0 = 0$ (2)

Theoretically, R is given by $\lim_{n\to\infty}\; R_n$ whrer tha sequence { $R_n$ }is defined by

$R_0 = 0$ (3)

$R_{n+1} = -A_0 A_1^{-1} - R_n^2 A_2 A_1^{-1}$ n ≥ 0 (4)

The sequence of $R_n$ is monotone sothat R could be evaluated from (3) and (4) by successive iteration.

Stability condition is given by

$\theta A_0\, e < \theta A_2\, e$ (5)

where θ is the invariant probability vector of the matrix A =$A_0 + A_1 + A_2$, given by

$$A = \begin{pmatrix} (\lambda + N\mu) & 0 & (\lambda + N\mu) \\ 0 & \beta & \beta \\ \nu(1 \quad p) & \nu p & \nu \end{pmatrix} \quad (6)$$

Ie, A is irreducible and

θ A = 0 (7)

θe = 1 (8)

$$[\theta_1\ \theta_2 \theta_3] \begin{pmatrix} (\lambda + N\mu) & 0 & (\lambda + N\mu) \\ 0 & \beta & \beta \\ \nu(1 \quad p) & \nu p & \nu \end{pmatrix} = [\,0 \quad 0 \quad 0\,]$$

$(\lambda + N\mu)\theta_1 + \nu(1 \quad p)\,\theta_3 = 0$ (9)

$\beta\theta_2 + \nu p\theta_3 = 0$ (10)

$(\lambda + N\mu)\theta_1 + \beta\theta_2 - \nu\theta_3 = 0$ (11)

$\theta_1 + \theta_2 + \theta_3 = 1$ (12)

$\theta_3 = 1 - \theta_1 \quad \theta_2$

solving

$(\lambda + N\mu)\theta_1 + \nu(1 \quad p)$[ 1-$\theta_1 \quad \theta_2$] =0

$\beta\theta_2$+ $\nu p$ [1-$\theta_1 \quad \theta_2$] = 0

[ + $\mu$ $\nu(1 \quad p)$ $\theta_1$+[ $\nu(1 \quad p)$] $\theta_2$ = $\nu(1 \quad p)$ (13)

$\nu p\theta_1 + (\beta + \nu p)\,\theta_2 = \nu p$ (14)

Solving

$\theta_1 = \frac{\beta\nu(1-p)}{(\lambda+N\mu)(\beta\theta \;\; + \nu p)+\beta\nu(1-p)}$ (15a)

From (9) , $\theta_3 = \frac{(\lambda+N\mu\ )}{\nu(1-p)}\ \theta_1$

Ie, $\theta_3 = \frac{\beta(\lambda+N\mu\ )}{(\lambda+N\mu\ )(\beta+\ \nu p)\beta\nu(1-p)}$ (15b)

From (10) $\theta_2 = \frac{\nu p}{\beta}\ \theta_3$

Ie, $\theta_2 = \frac{\nu p(\lambda+N\mu\ )}{(\lambda+N\mu\ )(\beta+\ \nu p)\beta\nu(1-p)}$ (15b)

$$(\theta_1\ \ \theta_2\ \theta_3)\begin{pmatrix}0 & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = [\theta_1 + \theta_2 + \theta_3]\begin{pmatrix}0 & 0 & l\mu\\ 0 & 0 & \beta\\ 0 & 0 & 0\end{pmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$\lambda\ (\theta_2 + \theta_3) < N\mu\theta_1 + \beta\theta_2$

$$\lambda[\frac{\nu p + \lambda\beta}{\beta}] < \frac{(N\mu + \nu p)}{(\lambda + N\mu\ )}$$

$\lambda[\frac{\nu p+\lambda\beta}{\beta}] < \nu$ as N→ ∞, which is the stability condition

Since R is a function of N, we shall write R(N) instead of R. Now, post multiplying (2) by e,

$R^2\ A_2e$ +R $A_1e$+ $A_0\,e$= 0

$A_0\,e$ = $R^2\ A_2e$ +R $A_1e$.

= R $A_2e$+ R[ $A_0\,e \quad$ R $A_2e$ ]

$A_0\, e \quad = \text{R}\, A_2 e$

Replace R by R(N)

Ie, $A_0\, e$ = R(N) $A_2 e$ (16)

$$\text{Ie, } \begin{pmatrix} 0 \\ \lambda \\ \lambda \end{pmatrix} = \text{R(N)} \begin{pmatrix} N\mu \\ \beta \\ 0 \end{pmatrix} \tag{17}$$

Evaluation of R(N):-

Using (3) and (4), we can numerical evaluate R. However, newts gave a better initial solution in order to expediate the convergence to R. such an initial solution is given by $R_0 = \eta$eu, where $\eta$ is the spectral radius and u is the curresponding normalized left eigen vector of R respectively. $\eta$ *and* $u$ can be computed without explicitly computing R, as follows:

Let Δ denote a diagonal matrix with elements equal to the diagonal elements of $A_1$

$$\text{Ie, } \Delta = \begin{pmatrix} (\lambda + N\mu) & 0 & 0 \\ 0 & (\lambda + \beta) & 0 \\ 0 & 0 & (\lambda + \nu) \end{pmatrix}$$

Now ,rearranging equation as R $\Delta = R^2 A_2 + R(A_1 + \Delta) + A_0$ (18)

R $\Delta = R^2 A_2\, \Delta^{-1} + R(A_1 \Delta^{-1} + \mathrm{I}) + A_0\, \Delta^{-1}$

Put $A_2\, \Delta^{-1} = B_2$ ; $A_1 \Delta^{-1} + I = B_1$ and $A_0\, \Delta^{-1} = B_0$

Then R = $R^2 B_2 + RB_1 + B_0$ (19)

Let $B(\mathfrak{z}) = B_2\, \mathfrak{z}^2 + B_1\, \mathfrak{z} + B_0$ (20)

Let x($\mathfrak{z}$) be thespetral radium for $0 \le \mathfrak{z} \le 1$.

Then, it can be shown that $\eta$ satisfies $\chi(\mathfrak{z}) = \mathfrak{z}$

By applying elementary procedures (such as bisection method) $\eta$ can be evaluated as the sort of the equation $\chi(\eta) = \eta$ in (0,1)

In this present problem, we can use the bisection to the evaluate $\eta$, that can be described as follows:

We restart with the lower and upper bounds say $\mathfrak{z}_1$, *and* $\mathfrak{z}_{2,}$ for the value of $\eta$. We then divide the interval ($\mathfrak{z}_{1,}\mathfrak{z}_{2,}$) at its mid point $\mathfrak{z}_3$ and determine whether $\eta$ lies in ($\mathfrak{z}_{1,}\mathfrak{z}_{3,}$,) or ( $\mathfrak{z}_{3,}\mathfrak{z}_{2,}$,) .

Elsners algorithm to evaluate the spectral radium is used to determine $\chi(\mathfrak{z}_3)$

(by newts). Then we obtain a tighter bounds for $\eta$, and using the tighter bounds, the bisection method is repeated. These interactions are continued until the bounds are closed enough for their midpoint to yield a sufficiently accurate estimate of μ. The initial value of $\mathfrak{z}_{2,}$ can be taken to be 1 because $\chi(1)$. The initial value of $\mathfrak{z}_{1,}$ can be taken as a small positive number. If $\chi(1)\leq \mathfrak{z}_{1,}$ we can set $\mathfrak{z}_{2,}$=$\mathfrak{z}_{3,}$

And try a smaller $\mathfrak{z}_{1,}$(say $\frac{\mathfrak{z}_{1,}}{2}$)

Choice of N :-

In this method the equilibrium possibilities of the states with i≥ N depend largely on $\eta(N)$(the spectral radius of R(N)). Therefore to minimize the effect of the approximation on the equilibrium possibilities, N must be chosen such that $\eta(N)$ is sufficiently close to $\eta(\infty)$. For this, a trail and error method is adopted. Starting with an initial value N. until $\eta$(N)- $\eta$ (∞) become a quantity smaller than a predetermined small value.

We develop a FORTRAN code for a perishable inventory model with the following assumptions. The demand process follows the Phase (PH) type distribution, replenishment follows Poisson distribution, and perishing time follows exponential distribution.

```
C                PROGRAM FOR G, MATRIX

        DIMENTION A0 (100,100),A1 (100,100), A2 (100,100), A11 (100,100)

        A12 (100, 100), H (100,100),XD(100, 100) ,XD0( 100, 100), XDI(100, 100) XD2(100, 100),
        XL(100, 100), U1(100,100), U2(100,100), U(100, 100), XM(100,100), ZO(100, 100),
        ZI(100,100), G1(100,100),G(100,100),T(100,100)

              WRITE (*,*)' ENTER THE ORDER OF THE MATRICES N'

              READ (*, 1)N

1             FORMAT (13)

              WRITE (*, *)' ENTER THE MATRIX A0'

              DO 10 I= 1, N

10 READ (*,2)  (A0 (I, J ) , J= 1 , N)

2             FORMAT (100F15.5)

              WRITE  (*,*)' ENTER THE MATRIX  A1

              DO 20 I= 1, N

20  READ (*,3)  (A1 (I, J ) , J= 1 , N)

3              FORMAT (100F15.5)

              WRITE  (*,*)' ENTER THE MATRIX  A2

              DO 30 I= 1, N

            30 READ (*,4)  (A2 (I, J ) , J= 1 , N)

4             FORMAT (100F15.5)

              DO 35 I= 1,N

              DO 35 J=1,1

              XD(I,J)=1.0

35             CONTINUE
```

```
        CALL SCALAR (A1, N, N, -1.0,A11)
        CALL XINVERSE (A11, N, A12)
         CALL MATRIXM (A12,A0, N, N, N, H)
        CALL MATRIXM (A12, A2, N, N, N, XL)
       DO 40 I=1,N
        DO 40 J= 1,N
      sXD(I, J)= XL(I,J)
40     CONTINUE
      DO 50 I=1, N
      DO 50 J=1, N
      T(I,J) = XL(I, J)
50    CONTINUE
200    CALL MATRIXM (H,XL, N, N, N, U1)
        CALL MATRIXM (XL, H, N, N, N, U2)
       CALL MATRIXMADD (U1, U2, N, N, U)
       CALL MATRIXM (H, H, N, N, N, XM)
      CALL XID (U, N,Z0)
      CALL XINVERSE (ZO,N, Z1)
      CALL MATRIXM (Z1, XM,N, N, N, H)
      CALL  MATRIXM(XL, XL, N, N, N, XM)
      CALL MATRIXM (Z1, XM, N, N, N, XL)
      CALL MATRIXM (T, XL, N,N,N,G1)
      CALL MATRIXADD (G, G1, N,N, G2)
      CALL MATRIM (T, H, N, N, N, T1)
       CALL MATRIXM (G, XD, N, N , 1,XD0)
      CALL SCALAR (XD0, N, -1.0,XD1)
      CALL MATRIXMADD (XD, XD1, N, 1, XD2)
```

```
                CALL XNORM (XD, XD1, N, 1, XD2)
                IF (XD3.GT.0.00000I) THEN
                DO 70 I =I,N
                DO 80 J=I,N
                G(I,J) = G2(I,J)
                T(I,J )= T1(I,J)
80              CONTINUE
70               CONTINUE
                GO TO 200
                END IF
                DO 60 I=1,N
                WRITE (1,5) (G2, I J) J = 1, N)
                FORMAT (100F15.5)
                CONTINUE
                STOP
                END

C                             PROGRAM MATRIX ADDITION
                  SUBROUTINE MATRIXADD (A, B, M, N, C)
                   DIMENSION
A(100,100), B(100,100), C(100,100), AI(100,100), B1(100,100)
                DO  4010  I= 1,M
                DO  4010 J= 1, N
                A1(I,J)= A (I,J)
                B1 (I, J)= B (I,J)
4010            CONTINUE
                DO  4030 I=1,M
```

```
              DO 4030 J=1,N
              4030 C(I,J ) = A (I, J ) + B ( I , J )
              RETURN
              END

C             PROGRAM FOR MATRIX MULTIPLICATION
              SUBROUTINE MATRIXM (A, B, M, N, C)
              DIMENSION
A(100, 100) B(100, 100) C(100, 100) , A(100, 100), B1(100,100)
              DO 3010 I= 1,M
              DO 3010 J=1,N
              A1(I,J ) =  A(I,J )
3010           CONTINUE
              DO 3020 I= 1,N
              DO 3020 J=1,L
              B1(I,J) = B (I,J )
3020          CONTINUE
              DO 3030 I=1,M
              DO 3030 J= 1, L
              C (I,J) = 0.0
              DO 3030 K= 1,N
3030          C (I,J )= C (I, J )+A (I,K)*B (K,J)
              RETURN
              END

C              PROGRAM FOR MATRIX INVERSE
              SUBROUTINE XINVERSE (A, N, B)
```

```
          DIMENSION A(100, 100) , B (100 ,100) , A1(100, 100)
          DO  1010I= J, N
          DO  1010 J = 1,N
          A1 (I,J ) =A (I, J )
1010   CONTINUE
          DO 1020 I= J, N
          DO  1030 J = 1,N
          B(I, J ) = 0.0
1030      CONTINUE
          B (I, I ) = 1.0
1020      CONTINUE
          DO 1040 K= J, N
          DO  1050 I= 1,N
          IF (I.EQ.K) GO TO 1050
          R=A (I,K )/ A(K,K )
          DO 1060 J= 1,N
          A(I,J ) =A  (I,J ) –R *A(K,J )
          B (I,J )= B (I,J )-R *B (K, J )
1060      CONTINUE
1050      CONTINUE
1040      CONTINUE
           DO 1070 J= 1,N
          DO 1080 J= 1,N
          B(I,J)= B (I,J )/ A (I,I )
1080      CONTINUE
1070       CONTINUE
          DO 1090 J= 1,N
```

```
            WRITE (*, 1003) (B(I, J ), J=1, N )
1003        FORMAT (*, 100F15.5)
1090        CONTINUE
            RETURN
            END

C            PROGRAM FOR SCALAR MULTIPLICATION OF A MATRIX
            SUBROUTINE SCALAR (A, M, N, Z, B)
            DIMENSION
A(100, 100) B(100, 100) C(100, 100) , A1(100, 100),
            DO 2010 I= 1,M
            DO 2010 J= 1,N
            A1(I,J)= A(I,J )
2010        CONTINUE
            DO 2020 I= 1,M
            DO 2030 J= 1,N
            B(I,J) = Z*A(I,J)
2030         CONTINUE
2020         CONTINUE
            RETURN
            END

C           PROGRAM FOR NORM OF A MATRIX
            SUBROUTINE XNORM (A, M, N, XE)
            DIMENSION A(100, 100) , A1(100, 100),
            DO 5010 I= 1,M
            DO 5010 J= 1,N
```

```
      A1(I,J)= A(I,J )
5010   CONTINUE
      BIG=ABS (A(1, 1))
      DO 5020 I= 1,M
      DO 5020 J= 1,N
      IF (BIG. LE. ABS (A(I,J))) BIG= ABS (A(I, J))
5020  CONTINUE
      XE= BIG
      RETURN
      END

C      PROGRAM FOR  I-A
      SUBROUTINE  XID (A,N, XI)
      DIMENSION  A(100, 100) ,XI(100, 100) ,  A1(100, 100),
      DO 6010 I= 1,N
      DO 6010 J= 1,N
      A1(I,J )=A(I,J)
6010   CONTINUE
      DO 6020 I= 1,N
      DO 6020 J= 1,N
      XI(I,J)=-A(I,J)
      XI(I,I)= 1.0-A(I,I)
6020   CONTINUE
      RETURN
      END
```

## Conclusion

The project was mainly proposed to study a perishable inventory model. The investigator aimed to control and design inventory models with perishable commodities using the performance measures thus obtained. The target was achieved for three general models and some of their variants.

In the chapter 2, we considered an (s,S) inventory model in which the perished items require a positive service time for repair. Stability of the system was fully analyzed. It was interesting to note that that process under consideration is a Level Independent Quasi Birth Death Process (LIQBD) and the underlying Markov chain is a quasi-Toeplitz matrix. We successfully obtained the product form solution. It is exactly a real life model as explained in the introduction of the chapter.

In chapters 3 and 4 we identify the inventory model with a special class of queues namely 'retrial queues'. We have considered the real life situations where the customers waiting for service in the 'virtual queue' undergoing decay. We have analyzed two situations: First, the server takes the perished customers from the virtual queue for repair ( some sort of a processed service). Second, Server goes for search of perishable customers in the virtual orbit there by trying to provide 'priority' for such customers. It is worthwhile to note here that the investigator successfully used the search mechanism introduced by himself and others [] for the effective utilization of a stochastic model. By the introduction of the search mechanism, the idle time of the server is significantly reduced and server utilization is optimally increased. Moreover, life time of the perishable commodities are optimally considered they have been 'picked' for service before perishing.

Though, most of the stochastic models are not analytically tractable, the investigator effectively used the techniques from computational probability to overcome this barrier. The investigator make use of the modern matrix geometric approximation for computing the steady state probabilities of the models considered. The presented algorithm for the calculation of the stationary state distribution under the given set of parameters of the model could be used for the evaluation of any performance measures of the system. Thus, it can be used for solving the problems of optimal service and retrial rates selection and search time for perishable commodities in the virtual queue.

We successfully developed a FORTRAN code a retrial queue with service follows Phase type (PH) distribution. The importance of Ph distribution is, it is dense in the class of all distributions. Thus, it will fit for any practical situation. Using the FORTRAN code developed, various performance measures can be obtained for real life models.

Two papers [28, 29] have already published as outcomes of the project and two papers are communicated.

## Bibliography


[1] Arrow K.A., S. Karlin and H.E.Scarf, Studies in the Mathematical Theory of Inventory and Production, Stanford University Press, Stanford, Calif, (1958).

[2] Artalejo.J.R., Accessible Bibliography on Retrial Queues, Mathematical and Computer Modelling, 30,1-6 (1999).

[3] Artalejo.J.R., A Classified Bibliography Research on Retrial Queues: Progress in 1990-1999,TOP,7,187-211,(1999).

[4] Artalejo J.R. and Falin G. I. Standard and Retrial Queueing Systems: A Comparative Analysis, *RevisitaMatematica Complutense,*15, (2002).

[5] Artalejo.J.R., V.C.Joshua and A. Krishnamoorthy, An M/G/1 Retrial Queue with Orbital Search by the Server, Advances in Stochastic Modelling,Notable Publications, New Jersey, Inc (2002).

[6] Berman O., and Kim E., stochastic Models for Inventory Management at Service Facilities, Comm.Stat-Stochastic Models, 15, 695-718,(1999).

[7] Bulinskaya, E., Some Results Concerning Optimal Inventory Policies. Theory Probabil. ItsAppl 9, 398-403, (1964).

[8] S. R. Chakravarthy, A. Krishnamoorthy and V. C. Joshua, Analysis of a Multi-Server Retrial queue with Search of Customers from the Orbit, Performance Evaluation, (2005).

[9] Cohen, M. A., Analysis of Single Critical Number Order Policies in Perishable Inventory Theory., Oppns Res., 24, 726-741. (1976).

[10] R. B. Cooper, Introduction to Queueing Theory, Elsevier, Amsterdam,(1981).

[11] Derman, C., and M.Klein. Inventory Depletion Management, Mgmt.Sci. 4, Pp 450-456, (1958)

[12]A. N. Dudin, A. Krishnamorthy, V. C. Joshua and G. Tsarenkov, Analysis of the BMAP/g/1 Retrial System withSearch of Customers from the Orbit, European Journal of Operational Research, (2004).

[13] Falin G. I.,A Survey of Retrial Queues, Queueing Systems7(2): 127-167,(1990).

[14] FalinG.I. and J.G.C.Templeton, Retrial Queues, Chapman and Hall, London (1997).

[15] Hadley, G., and Whitin.T.M., Analysis of Inventory Systems, Prentice Hall. Inc.Engkwodchits,New Jersey, (1963).

[16] Hughes, R.I.G, Models and Representation, Philosophy of Science, 64:, 325-336 (1997).

[17] A. Krishnamoorthy, T.G.Deepak and V.C.Joshua, An M/G/1 Retrial Queue with Non-Persistent Customers and Orbital search, Stochastic Analysis and Applications,23, 975-997,(2005).

[18] A. Krishnamoorthy, Viswanath C., Narayanan, T.G.Deepak and Vineetha K., Control Policies for Inventory with Service Time, Stoch. Anal. Appl., 24, no. 4, 889-900, (2006).

[19] Leng, Mary, Mathematics and Reality, Oxford, 2010.

[20] Morgan,, Mary, Learning from Models, in Morgan and Morrison, 347-388,(1999).

[21] Millard D.W., Industrial Inventory Models as Applied to the Problem of Inventorying the Whole Blood, Industrial engineering Analysis of A Hospital blood Laboratory, Engineering Experimental Station Bulletin 180, Ohio State, University, Columbus, (1960).

[22] Morrison, Margaret, Modelling Nature: between Physics and the Physical World, Philosophia Naturalis, 35: 65-85, (1998).

[23] Nahmias S., Inventory Depletion Management when the Field Life is Random., Mgmt Sci.20, 1276-1283, (1974).

[24] M.F.Neutes, Structural Stochastic Matrices of M/G/1 type and their applications, Marcel decker, New York, (1989).

[25] M.F.Neutes, Matrix – Geometric Solutions in Stochastic Models, An Algorithmic Approach, The John Hopkins University Press, Baltimore,MD, (1984).

[26 ] Neutes M.F., and M.F.Ramalhoto, A Service Model in which the Server is Requested to Serach for Customers, Journal of Appl. Prob., 21: 157-166, (1984)

[27] V. Ramaswami and G. Latoushe, introduction to Matrix Analytic Methods in Stochastic Models, SIAM and ASA, (1999).

[28] Varghese C. Joshua, Models: The Building Blocks of Science, New Frontiers in Education, Pp76-83, Jan-Mar (2015).

[29] Varghese C. Joshua, Mathematical Models: An Unreasonable Effectiveness of Mathematics in Natural Sciences, Proceedings of National Conference, Assumption College, (2015).

[30] Wigner, E, The Unreasonable Effectiveness of Mathematics in the Natural Sciences, Communications on Pure and applied Mathematics, 13: 1-14, (1960)

[31] Yang, T., and J.G.C. Templeton, A survey on Retrial Queues, Queueing Systems,2: 201-233, (1987).

[32] Yen, H., Inventory Management for a Perishable Product Multi-echilon System Ph.D Dissertation, North Western University, Evanston, (1965).